Class 2-9 Notes.

Calc AB Ch 2B/3 Test Practice

Name: 11/5/21 key -Updated 11/10/21 & 11/13/21 Block: Seat:

email possible errata to cthiel@sfhs.net

Chapter 2B skills Check List:

1 ...... Related Rates Word Problems (Section 2.6).

Chapter 3 skills Check List:

1 ...... Absolute (p166) and Relative Extrema (p167)

2 ...... Critical Numbers (168)

3 ...... Extreme Value Theorem (EVT) (p166)

4 ...... Candidates Test (p 169)

5 ...... Rolle's Theorem (p 174)

6 ...... Mean Value Theorem (MVT) for rates (p 176)

7 ...... Increasing and Decreasing (181)

8 ...... 1^{st} Derivative Test for Relative Extrema (p 183)

9 ...... Concavity (p. 192-193)

10 ...... Points of Inflection (POI) (p.193-194)

11 ...... 2^{nd} Derivative Test for Relative Extrema (p 195)

12 ...... Limits at $\pm\infty$ (end behavior) (p 199-200, 205)

13 ...... Horizontal Asymptote (p. 200)

14 ...... Curve Sketching (Section 3.6)

15 ...... Optimization Word Problems (Section 3.7)

Delta Math Check List:

1 ...... Practice Related Rates (4 skills)

2 ...... Practice EVT (2 skills)

3 ...... Practice MVT (3 skills)

4 ...... Practice Function Analysis (3.3) (6 skills)

5 ...... Practice Function Analysis (3.4) (6 skills)

Khan Academy Check List:

1 ...... Contextual applications of Derivatives Unit topic: Solving Related Related Rates Problems (AP Unit 4.4)

2 ...... Applying Derivatives to Analyze Functions Unit (AP Unit 5)

1. (Calculator NOT Active) Use the Candidates Test to identify the absolute extrema of $f(x) = x^3 - 6x^2 + 9x + 5$ on the interval $0 \le x \le 4$.

$f'(x) = 3x^2 - 12x + 9$

$0 = 3(x^2 - 4x + 3)$

$0 = 3(x-1)(x-3)$

EP: $x = 0$, $x = 4$

CP: $x = 1$, $x = 3$

x	0	1	3	4
$f(x)$	5	9	5	9

Abs. Max on $[0,4]$ is 9

Ab Min on $[0,4]$ is 5

by the Candidates Test

2. (Calculator NOT Active) Use the Candidates Test to identify the absolute extrema of $f(x) = x + \dfrac{7}{x}$ on the interval $1 \le x \le 3$.

E.P.: $(1, 8)$

$(3, 3 + \frac{7}{3})$

$f'(x) = 1 - \frac{7}{x^2}$

$0 = 1 - \frac{7}{x^2}$

$\frac{7}{x^2} = 1$

$x = \pm\sqrt{7}$, only $\sqrt{7}$ on $[1,3]$

CP: $(\sqrt{7}, \sqrt{7} + \frac{7}{\sqrt{7}})$

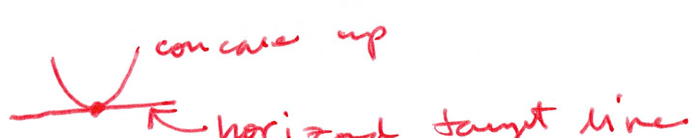
(0 not on $[1,3]$)

x	1	$\sqrt{7}$	3
$f(x)$	8	$\sqrt{7} + \frac{7}{\sqrt{7}}$	$\frac{16}{3}$ (5.333)

Ab max is 8, Ab Min $2\sqrt{7}$ (5.292) by Candidates Test

3. Let f be a twice differentiable function. If $f'(7) = 0$ and $f''(7) > 0$ what conclusion can be made and why?

$f(7)$ is a relative min by 2^{nd} Der Test



4. (Calculator NOT Active) Find at least one c such that the Mean Value Theorem applies to the function $f(x) = x^3$ on the interval $[0, 1]$ and write the equation(s) of the tangent line(s) to the curve at $x = c$.

$$\text{AROC} = \frac{f(1) - f(0)}{1 - 0} = \frac{1-0}{1} = 1 = f'\left(\frac{\sqrt{3}}{3}\right)$$

because

$f'(x) = 3x^2$

$1 = 3x^2$

$x = \pm\sqrt{\frac{1}{3}}$, only $\frac{\sqrt{3}}{3}$ is on $[0, 1]$

tangent line @ $x = \sqrt{3}/3$:

$y = 0$ (or) $y - 0 = 1(x - 0)$ (or) $y - 1 = 1(x - 1)$

5. (Calculator Active) Find all c on $[0, 2]$ such that the Mean Value Theorem applies to the function $f(x) = x^2 + 3x - 4\sin(2x + 3)$ on the interval $[0, 2]$.

$$\text{AROC} = \frac{f(2) - f(0)}{2 - 0} = 3.968 = f'(0.909)$$

Since

$f'(x) = 2x + 3 - 8\cos(2x + 3) = 3.968$

$x = 0.909$

6. Find absolute extrema for the function $f(x) = x^3 - 3x + 2$ on the interval $[-3, 2]$. Justify your conclusion.

use Cand. test $\quad f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

E.P: $(-3, f(-3))$ $\quad$ C.P.: $(-1, f(-1))$

$(2, f(2))$ $\quad$ $(1, f(1))$

x	-3	-1	1	2
$f(x)$	-16	4	0	4

Ab. min is -16

Ab max is 4

By the candidates test

(Calc not Active)

7. Let $f(x) = x^3 - 3x + 2$.

(a) Find f' and f'',

$f'(x) = 3x^2 - 3 = 3(x^2-1)$

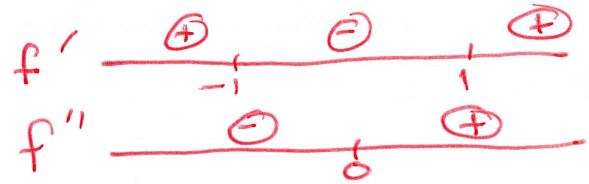
$f''(x) = 6x$

(b) Find any and all of the critical points of f. f' and f''

$f'(x) = 3(x-1)(x+1) = 0$ cp ± 1

$f''(x) = 0 = 6x$ cp 0

(c) Draw sign lines for f' and f''



(d) On what intervals are f increasing and decreasing? Justify.

inc when $f'(x) > 0$: $(-\infty, -1)$, $(1, \infty)$

dec when $f'(x) < 0$: $(-1, 1)$

(e) On what intervals are f concave up or down? Justify.

up : when $f''(x) > 0$ $(0, \infty)$

down when $f''(x) < 0$ $(-\infty, 0)$

(f) Find all relative extrema and justify your conclusions with the 1^{st} Derivative Test

$f'(x)$ changes from pos to neg @ $x = -1$, $f(-1)$ rel max

$f'(x)$ Δ's from neg to pos @ $x = 1$, $f(1)$ rel min

(g) Find all relative extrema and justify your conclusions with the 2^{nd} Derivative Test

$f'(-1) = 0$, $f''(-1) < 0$ concave down, $f(-1)$ rel max

$f'(1) = 0$, $f''(1) > 0$ concave up, $f(1)$ rel min

(h) If any exist, find any points of inflection (POI). Justify.

$(0,2)$ is a P.O.I because $f''(0)$ changes sign

(i) What is the end behavior of f (That is, find $\lim_{x \to \pm\infty} f(x)$)

$\lim_{x \to \infty} f(x) = \infty$ $\lim_{x \to -\infty} f(x) = -\infty$

(odd degree leading term pos.)

8. (Calculator NOT active) Given the function $g(x) = \dfrac{4x}{x^2+1}$, its 1^{st} derivative $g'(x) = \dfrac{4(1-x^2)}{(x^2+1)^2}$ and its 2^{nd} derivative $g''(x) = \dfrac{8x(x^2-3)}{(x^2+1)^3}$

(a) Make sign lines for g, g', and g''

g: $\ominus$ | $\oplus$; 0

g': $\ominus$ | $\oplus$ | $\ominus$; -1, 1

g'': $\ominus$ | $\oplus$ | $\ominus$ | $\oplus$; $-\sqrt{3}$, 0, $\sqrt{3}$

(b) Find the (x, y) coordinates of any intercepts, and graph them.

$(0,0)$

(c) Find any vertical or horizontal asymptotes and g's end behavior (i.e.: $\lim_{x\to\pm\infty} g(x)$) and use this to improve the graph.

$\left.\begin{matrix} n=1 \\ m=2 \end{matrix}\right\}$ $n<m$ so $\lim_{x\to\pm\infty} g(x) = 0$ horiz. asymptote $y=0$

(d) Find the relative extrema and indicate them on the graph with a small horizontal bar through any extrema.

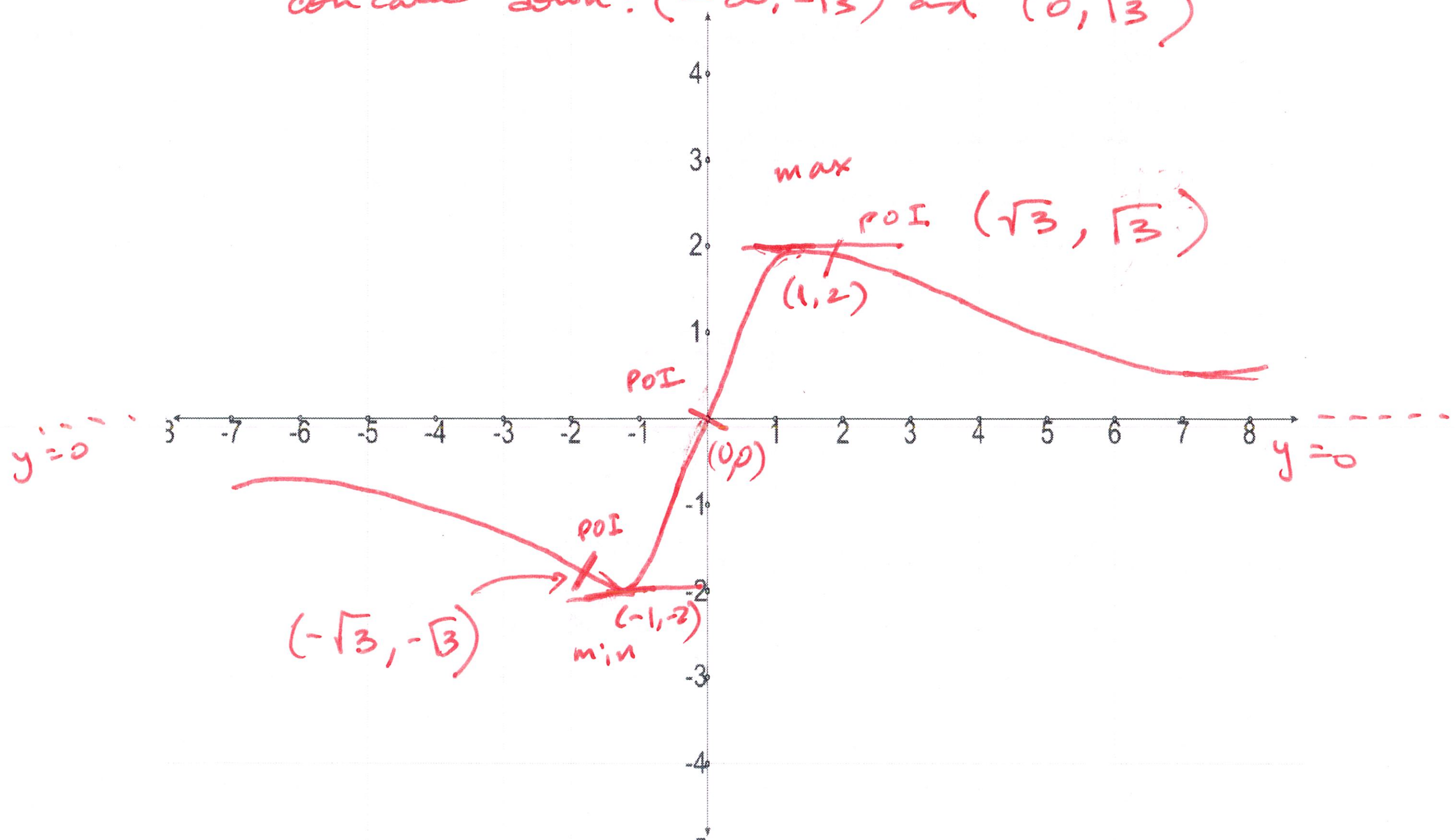
$f'(-1)=0$, $f''(-1)>0$ concave up so $(-1,-2)$ rel min.

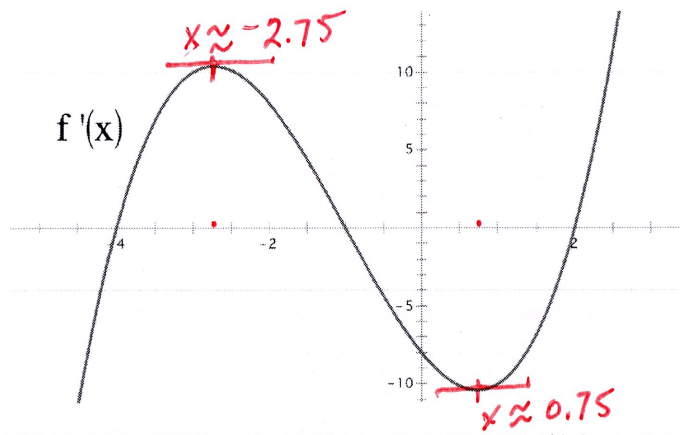
$f'(1)=0$, $f''(1)<0$ concave down so $(1,2)$ rel max

(e) Improve the graph by noting where it is concave up or concave down, and find any points of inflection. Indicate a POI on the graph with a small perpendicular bar through the point.

concave up: $(-\sqrt{3},0)$ and $(\sqrt{3},\infty)$

concave down: $(-\infty,-\sqrt{3})$ and $(0,\sqrt{3})$





9. (Calculator NOT active) Given the graph of $f'(x)$ above:

(a) Approximate the sign lines for $f'(x)$ and $f''(x)$.

f': $\ominus$ | -4 | $\oplus$ | -1 | $\ominus$ | 2 | $\oplus$

f'': $\oplus$ | $\frac{-11}{4}$ | $\ominus$ | $\frac{3}{4}$ | $\oplus$

(b) Find a relative maximum of f and use the 1^{st} Derivative Test to justify your conclusion.

$f'(x)$ changes from pos to neg. at $x = -1$ so $f(-1)$ is a rel. max (1st Der. Test)

(c) Find a relative minimum of f and use the 2^{nd} Derivative Test to justify your conclusion.

$f'(-4) = 0$ and $f''(-4) > 0$, concave up, so $f(-4)$ is rel. min

(or) $f'(2) = 0$ and $f''(2) > 0$, concave up, so $f(2)$ is a rel. min

(d) On what interval(s) is f increasing? Justify.

$-4 < x < -1$ and $x > 2$ $(-4, -1)$; $(2, \infty)$

because $f'(x) > 0$ (positive)

(e) On what interval(s) is f decreasing? Justify.

$x < -4$, $-1 < x < 2$ $(-\infty, -4)$, $(-1, 2)$

because $f'(x) < 0$ (neg)

(f) On what interval(s) is f concave up? Justify.

$x < \frac{-11}{4}$ and $x > \frac{3}{4}$ $(-\infty, \frac{-11}{4})$, $(\frac{3}{4}, \infty)$

because $f''(x) > 0$ (pos)

(g) On what interval(s) is f concave down? Justify.

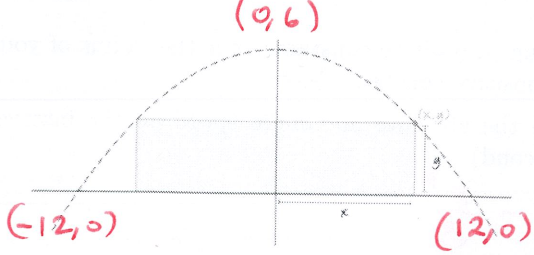
$\frac{-11}{4} < x < \frac{3}{4}$ $(\frac{-11}{4}, \frac{3}{4})$ because $f''(x) < 0$ (neg)

(h) Find all the points of inflection of f, and justify why it is a point of inflection.

$(\frac{-11}{4}, f(\frac{-11}{4}))$ and $(\frac{3}{4}, f(\frac{3}{4}))$ are P.O.I.'s

because $f''(x)$ changes signs there.

10. (Calculator NOT Active) A rectangle has its bottom edge on the x axis and its top corners are on the the graph $y = 6 - \dfrac{x^2}{24}$ What length and width should the rectangle have so that the rectangle has the largest area?



length $= 2x$

width $= y = 6 - \dfrac{x^2}{24}$

$A(x) = 2x\left(6 - \dfrac{x^2}{24}\right)$

$A(x) = 12x - \dfrac{x^3}{12}$

$A'(x) = 12 - \dfrac{x^2}{4}$ so $x^2 = 4 \cdot 12$, $x = 4\sqrt{3}$

Plausible Domain : $0 < x < 12$

A' 0 $\oplus$ $\frac{1}{2}$ $\ominus$

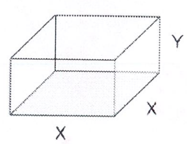
(or)

Candidates Test: $A'(x)$ Δs pos to neg @ $x = 2$ (loc. Max)

x	0	$4\sqrt{3}$	12
$A(x)$	0	$32\sqrt{3}$	0
		Ab Max	

The largest Area is when the length is $8\sqrt{3}$ and the width is 4.

11. (Calculator NOT Active) Engineers are designing a box-shaped aquarium with a square bottom and an open top. The aquarium must hold 500 cubic feet of water. What dimensions should they use to create an acceptable aquarium with the least amount of glass?



x = side of square base

y = height of quarium $= \dfrac{500}{x^2}$

Volume $= x^2 y$

$500 = x^2 y$

$y = \dfrac{500}{x^2}$

Candidate Test

x	0	10	$\sqrt{500}$
$A(x)$	0	300	$500 + 40\sqrt{5}$
		Min	

Area $= 4x(y) + x^2 =$

$A(x) = 4x\left(\dfrac{500}{x^2}\right) + x^2 = \dfrac{2000}{x} + x^2$

$A'(x) = -\dfrac{2000}{x^2} + 2x = 0$

$2x = \dfrac{2000}{x^2}$ CP = 10

$x^3 = 1000$ EP = 0, $\sqrt{500}$

$x = 10$

The least amount of glass is when the base is $10' \times 10'$ and the height is $5'$

12. In 1977 P.M. Tuchinsky wrote an article called "The Human Cough". In it, the speed s of the air leaving your windpipe as you cough can be modeled by the function

$$s(x) = kx^2(R - x)$$

where R is radius of your windpipe (trachea) while resting (a positive constant), x is the radius of your windpipe while coughing (a positive variable), and k is positive constant. for

If for a particular person $k = \frac{1}{3}$ and $R = 27$ mm, find the absolute maximum speed on the interval $0 \le x \le 27$. (Your answer will be in terms of mm per second)

$s(x) = \frac{x^2}{3}(27-x) = 9x^2 - \frac{x^3}{3}$

$s'(x) = 18x - x^2$

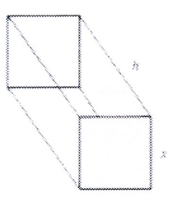
$0 = x(18-x)$

Candidates test:

x	0	18	27
$s(x)$	0	$\underbrace{972}_{\text{max}}$	0

The max speed of the air leaving a cough is 972 mm per second.

13. A box has 2 square ends and 4 rectangular sides. The square ends are made out of plastic that costs \$5 dollars per square foot, and the cardboard sides cost \$3 dollars per square foot. Find the dimensions of the box that has a volume of 60 cubic feet, that is the cheapest to make. *Hint:* Use the candidates test, first derivative test, or second derivative test to justify your claim that it is indeed the minimum cost.



x = side of square base of box

h = height (or length) of side of box

$V = x^2 h$

$60 = x^2 h$, so $h = \frac{60}{x^2}$

$C(x,h) = \$5(2x^2) + \$3(4xh)$

$C(x) = 10x^2 + 12x\left(\frac{60}{x^2}\right)$

$C(x) = 10x^2 + \frac{720}{x}$

$C'(x) = 20x - \frac{720}{x^2} = 0$

$20x = \frac{720}{x^2}$

$x^3 = 36$

$x = 36^{1/3} \approx 3.301$ in

3.302 in

Area of two bases: $2x^2$

Area of sides: $4xh$

Domain: $0 < x < \sqrt{60}$

c': 0 — (−) — $36^{1/3}$ — (+) — $\sqrt{60}$

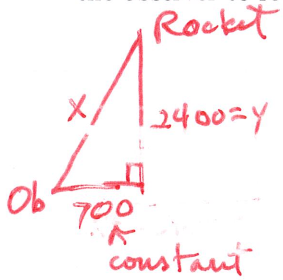
$c'(x)$ Δ's from neg to pos at $x = 36^{1/3}$ feet.

Candidates:

x	0	$36^{1/3}$	$\sqrt{60}$
$C(x)$	∞	327.08	692.95

The min cost is when the base is $36^{1/3}$ in $\times$ $36^{1/3}$ in and the sides are $\frac{60}{36^{2/3}}$ (5.503 in)

14. An observer stands 700 ft away from a launch pad to observe a rocket launch. The rocket blasts off and maintains a velocity of 900 ft/sec. Assume the scenario can be modeled as a right triangle. How fast is the observer to rocket distance changing when the rocket is 2400 ft from the ground?



① Given $\frac{dy}{dt} = 900$ ft/sec

② find $\left.\frac{dx}{dt}\right|_{\substack{y=2400 \\ x=2500}} = ?$

③ $y^2 + 700^2 = x^2$ (pythagorean Th)

④ $2y\frac{dy}{dt} + 0 = 2x\frac{dx}{dt}$

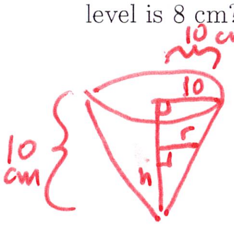
⑤ $2(2400)(900) = 2(2500)\frac{dx}{dt}$

$\frac{dx}{dt} = 864$ ft/sec

The distance from the observer to the rocket is increasing at a rate of 864 feet per second.

15. A conical paper cup is 10 cm tall with a radius of 10 cm. The cup is being filled with water so that the water level rises at a rate of 2 cm/sec. At what rate is water being poured into the cup when the water level is 8 cm?

Vol of



$\frac{r}{10} = \frac{h}{10}$

so $r = h$

① given $\frac{dh}{dt} = 2$ cm/sec

② Find $\left.\frac{dV}{dt}\right|_{h=8\text{cm}} = ?$

③ $V = \frac{\pi r^2 h}{3} = \frac{\pi h^3}{3}$

④ $\frac{dV}{dt} = \frac{3\pi h^2}{3}\frac{dh}{dt}$

⑤ $\frac{dV}{dt} = \pi 8^2(2) = 128\pi$ cm³/sec

The water is being added at a rate of 128π cubic cm per second